Existence of (F,Y) bounded solutions of linear Sylvester system of first order differential equations

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Abstract

In this paper, we establish (F,Y) bounded solutions of linear first order Sylvester system of matrix differential equation and deduce the existence of Y-bounded solution as a particular case.

Keywords: Sylvester linear systems, Lyapunov systems, Y-bounded solutions, Stability and asymptotic stability, Fundamental matrix solutions, Variation of parameters formula

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1. Introduction

Sylvester system of first order linear non-homogeneous equation is an interesting area of current research and the general form of its solution in two fundamental matrices is only established by Murty and Prasad in the year of 1989 [9]. The paper attracted many eminent mathematicians like Richard Bellman, Don Fausett, Lakshmikantham to mention a few. Recent results established byViswanadh, V. Kanuri, *et. al.*, is the main motivation behind our results. The concept of Y-bounded solutions for linear system of differential equations is due to T. G. Halam [14]. The novel idea adopted by Viswanadh, Wu and Murty [8] on the existence of $(F \ddot A Y)$ bounded solutions and on the existence of Y-bounded solutions by Kasi Viswanadh, V. Kanuri, *et. al*. [4-7,11,12,13] on time scale dynamical systems is a useful and significant contribution to the theory and differential and difference equations. Further these ideas have been extended by Kasi Viswanadh V. Kanuri to fuzzy differential equations in a novel concept, and is very interesting and useful contribution to the theory of differential equations and also in applications to control systems. The results established on stability, controllability criteria established on state scale dynamical systems on first

order linear systems [9] can be generalized to (F, Y) bounded solutions to Sylvester linear system of differential equations. This paper is organized as follows: section 2 presents a criterion for the existence of F -bounded solution of the matrix linear system $T' = AT$ and Y -bounded solution of the linear system $T' = B^*T$ (where * refers to the transpose of the complex conjugate). By super imposing these two solutions, we establish the general solution of the linear matrix Sylvester system

$$T' = A(t)T + TB(t) \tag{1.1}$$

where T is a square matrix of order (n ´ n) and $A(t)$, $B(t)$ are also n ´ n matrices. We present our basic results that are available in literature [4, 5, 6, 7, 8, 10, 11]. Our main results are established in section 3. This section also presents criteria for the Sylvester system (1.1) to be stable, asymptotically stable, and establishing the result on controllability. Throughout this paper, $Y(t)$ stands for a fundamental matrix solution of the linear system

$$T' = A(t)T \tag{1.2}$$

and $Z(t)$ stands for a fundamental matrix solution of the linear system

$$T' = B^*T . \tag{1.3}$$

2. Preliminaries

In this section, we shall be concerned with establishing general solution of the Sylvester linear system and present FY -bounded solution of the linear system (1.2) and then YZ -bounded solution of the system (1.3).

Theorem 2.1 T is a solution of (1.1) if and only if $T = YCZ^*$, where C is a constant square matrix and Y is a fundamental matrix solution of (1.2) and Z is a fundamental matrix solution of (1.3).

Proof: It can easily be verified that T defined by YCZ^* is a solution of (1.1). For

$$T' = Y'CZ^* + YCZ^{*'} = A(t)YCZ^* + YCZ^*B = AT + TB .$$

Hence, YCZ^* is a solution of (1.1). Now, to prove that every solution is of this form, let T be a solution, and K be a matrix defined by $K = Y^{-1}T$. Then, $Y'K + YK' = AYK + YKB^*$ or $YK' = YKB^*$ or $K' = KB^*$ or K*' = BK*. Since Z is a fundamental matrix solution of (1.3), it follows that there exists a constant square matrix C such that $K^* = ZC^*$ or $K = CZ^*$. Since $T = YK = YCZ^*$. □

In [4], Kasi Viswanadh, V. Kanuri, *et. al.* presented a novel concept on Y -bounded solutions of linear differential systems on time scales. We use these ideas as a tool to establish (F, Y) bounded solutions of the Sylvester system (2.1).

Definition 2.1 A function $Y:\mathbb{R}^{+}\to\mathbb{R}^{n^2}$ is said to be F-bounded solution on $\mathbb{R}$ if FY is bounded on $\mathbb{R}$.

Definition 2.2 A function $Z:\mathbb{R}^{+}\to\mathbb{R}^{n^2}$ is said to be Y-bounded solution on $\mathbb{R}$ if $Z^{*}Y^{*}$ is bounded on $\mathbb{R}$.

Definition 2.3 A function $Y:\mathbb{R}^{+}\to\mathbb{R}^{n^2}$ is said to be F-Lebesgue integrable on $\mathbb{R}^{+}$ if $Y(t)$ is measurable and $F(t)Y(t)$ is Lebesgue integrable on $\mathbb{R}^{+}$.

Definition 2.4 A function $Z:\mathbb{R}^{+}\to\mathbb{R}^{n^2}$ is said to be Y-Lebesgue integrable on $\mathbb{R}^{+}$ if $Z^{*}(t)$ is measurable and $Z^{*}(t)Y^{*}(t)$ is Lebesgue integrable on $\mathbb{R}^{+}$.

Let $F_i:\mathbb{R}^{+}\to\mathbb{R}^{n}$, $i=1,2,...,n$, be continuous and let $F(t)=(F_1(t),F_2(t),...,F_n(t))$ be linearly independent so that F is invertible and also we assume Y is invertible.

By a solution of the linear system (1.1), we mean $Y(t)CZ^{*}(t)$, which is an absolutely continuous function and satisfies (1.1) for almost all $t^3 \ 0$.

Let Y be a fundamental matrix solution of (1.2) satisfying $Y(0)=I_n$ and Z be a fundamental matrix solution of (1.3) satisfying $Z(0)=I_n$. Let X_1 denote the subspace of $\mathbb{R}^{n}$ consisting of all vectors whose values are of F-bounded solutions of (1.2) for $t=0$ and X_2 be the arbitrary fixed subspace of $\mathbb{R}^{n}$ supplementary to X_1. Further, let P_1 be the projection matrix of $\mathbb{R}^{n}$ onto X_1 ($P_1^2=P_1$ and $P_1:\mathbb{R}^{n}\to\mathbb{R}^{n}$) and let $P_2=I-P_1$ be the projection matrix on X_2

Definition 2.5 A function $f:\mathbb{R}\to\mathbb{R}^{n'n}$ is said to be F-bounded on $\mathbb{R}$ if $F(t)f(t)$ is bounded on $\mathbb{R}$, i.e. $\sup_{t\hat{I}\mathbb{R}}\|F(t)f(t)\|<¥$.

Definition 2.6 A matrix $Y:\mathbb{R}\to\mathbb{R}^{n'n}$ is said to be F-bounded on $\mathbb{R}$ if the matrix $F(t)Y(t)$ is bounded on $\mathbb{R}$, i.e. there exists an $M>0$ such that $\sup_{t\in R}\|\Phi(t)Y(t)\|\le M$

Definition 2.7 A matrix $Y:\mathbb{R}\to\mathbb{R}^{n'n}$ is said to be F-integrable on $\mathbb{R}$ component-wise if $F(t)Y(t)$ is integrable on $\mathbb{R}$, i.e. $\int_{0}^{\infty}\|\Phi(t)Y(t)\|dt<\infty$.

Definition 2.8 A matrix function $(F,Y):\mathbb{R}\to\mathbb{R}^{n^{2'} \ n^2}$ is said to be (F,Y)-bounded if the matrix $\|FYZ^{*}Y^{*}\|$ is bounded on $\mathbb{R}$.

3. Main results

In this section, we shall be concerned with the existence of (F, Y)-bounded solution of the linear Sylvester system (1.1), and then present the stability and asymptotic stability of the Sylvester system.

Theorem 3.1 Let A and B be $(n' \; n)$ continuous square matrices on $¡$. Then, the system (1.1) has at least one (F, Y)-bounded solution on $¡$ for every continuous (F, Y)-bounded function if and only if there exists a positive constant K such that

$$\grave{\mathrm{O}}_{-¥}^{¥} \left\| F(t)Y(t)PZ^*(t)Y^*(t) \right\| £ \; K \quad \text{for all } t ³ \; 0 \tag{3.1}$$

where $P = P_-$ on $(-¥, t)$, $P = P_0 + P_+$ on $(t, 0)$, $P = P_+$ on $(0, ¥)$, $P = P_-$ on $(-¥, 0)$, $P = P_0 + P_-$ on $(0, t)$, $P = P_+$ on $(t, ¥)$.

Proof: First, suppose the linear Sylvester system has at least one (F, Y)-bounded solution on $¡$ for every continuous (F, Y)-bounded function on $¡$. Then, it is claimed that there exists a constant $K > 0$ such that the inequality (3.1) holds. Let B be the Banach space of all (F, Y)-bounded continuous functions $T : ¡ \; ® \; ¡ \; {}^{n^2}$ with norm $\|T\|_{\mathrm{B}} = \sup\limits_{t Î ¡} \left\| F(t)Y(t)Z^*(t)Y^*(t)T(t) \right\|$, we define

(i) C: the Banach space of all (F, Y)-bounded continuous functions $T : ¡ \; ® \; ¡ \; {}^{n^2}$ with norm

$$\|T\|_C = \sup\limits_{t\in\mathbb{R}} \|\Phi(t)Y(t)Z^*(t)\Psi^*(t)\|$$

(ii) $\mathcal{B}$: the Banach space of all (F, Y)- V integrable functions $T : ¡ \; ® \; ¡ \; {}^{n^2}$ with norm

$$\|\mathrm{T}\|\mathcal{B} = \int_{-\infty}^{\infty} \|\Phi(t)Y(t)Z^*(t)\Psi^*(t)\| dt$$

(iii) D: the set of all continuous function $T : ¡ \; ® \; ¡ \; {}^{n^2}$ which are absolutely continuous on all intervals $J Ì ¡$, (F, Y)-bounded on $¡$, $T(0) Î X_- Ä X_+$, and $\boldsymbol{T'} = \boldsymbol{AT} + \boldsymbol{TB} \in \mathcal{B}$.

Step 1: We first claim that $(D, \| \; \|_D)$ is a Banach space. For, we first note that $(D, \| \; \|_D)$ is a vector space. Let $\{T_n\}_{n Î ¥}$ be a fundamental sequence in $\mathcal{B}$. Then, there exists a continuous (F, Y)-bounded function on $¡$ such that

$$\lim\limits_{n ® ¥} F_n(t)T_n(t)Y_n^*(t)T_n^*(t) = F(t)T(t)Y^*(t)T^*(t)$$

uniformly on $¡$. From the inequality

$$\|T_n(t) - T(t)\|$$
$$\leq \|\Phi^{-1}(t)\|\{\|\Phi(t)\Psi^*(t)\Psi^{*^{-1}}(t)\|\|\Psi^*(t)\|$$
$$- \|\Phi^{-1}(t)\|\|\Phi(t)T(t)\Psi^*(t)\|\|\Psi^{*^{-1}}(t)\|\}$$

Hence, $\lim_{n \to \infty} T_n(t) = T(t)$ uniformly on every compact subset of $¡$. Thus, $T(0) \hat{I} \ X_- \ \ddot{A} \ X_+$. Thus, $(D, \| \ \|_D)$ is a Banach space.

We now establish variation of parameters formula for the non-homogeneous Sylvester system

$$T' = A(t)T + TB(t) + F(t) \tag{3.2}$$

where $F(t)$ is a given $(n' \ n)$ square matrix. Let T be any solution of (3.2) and $\overline{T}$ be a particular solution of (3.2). Then $T - \overline{T}$ is a solution of the homogenous system (1.1). Any solution of the homogeneous system is of the form $T(t) = Y(t)CZ^*(t)$, where $Y(t)$ is a fundamental matrix solution of (1.2) and $Z(t)$ is a fundamental matrix of (1.3). Such a solution cannot be a solution of (3.1) unless $F(t) = 0$.

We seek a particular solution of (3.1) in the form

$$\overline{T}(t) = Y(t)C(t)Z^*(t)$$

and see that $\overline{T}(t)$ is a particular solution of (3.1). Now,

$$\overline{T}'(t) = Y'(t)C(t)Z^*(t) + Y(t)C'(t)Z^*(t) + Y(t)C(t)Z^{*'}(t)$$
$$= A(t)Y(t)C(t)Z^*(t) + Y(t)C'(t)Z^*(t) + Y(t)C(t)Z^*B$$

Now on substitution in the eqn (3.1) gives

$$A(t)Y(t)C(t)Z^*(t) + Y(t)C'(t)Z^*(t) + Y(t)C(t)Z^*B$$
$$= A(t)Y(t)C(t)Z^*(t) + Y(t)C(t)Z^*(t)B(t) + F(t)$$

which gives $Y(t)C'(t)Z^*(t) = F(t)$, then $C'(t) = Y^{-1}(t)F(t)Z^{*^{-1}}(t)$, or

$$C(t) = \int_a^t Y^{-1}(s)F(s)Z^{*^{-1}}(s) \ ds$$

and hence, $\overline{T}(t) = Y(t)\int_a^t Y^{-1}(s)F(s)Z^{*^{-1}}(s) \ ds \ Z^*(t)$. Now, it can easily be verified that $\overline{T}(t)$ is a solution of (3.2), and the general solution is given by

$$T(t) = Y(t)CZ^*(t) + \overline{T}(t) = Y(t)CZ^*(t) + Y(t)\grave{O}_a^t Y^{-1}(s)F(s)Z^{*-1}(s)\, ds\, Z^*(t).$$

We now claim that three exists a constant $K_0 > 0$ such that for every $F \in \mathscr{B}$ and for corresponding solution of $T \hat{I}\ D$ of (3.2), we have

$$\sup_{t\hat{I}\ ¡}\left\|F(t)T(t)Y^*(t)\right\| £\ K_0 \sup_{t\hat{I}\ ¡}\left\|F(t)F(t)Y^*(t)\right\|$$

or

$$\sup_{t\hat{I}\ ¡} \max_{1£\ i£\ n}\left\|F_i(t)T_i(t)Y_i^*(t)\right\| £\ K_0 \sup_{t\hat{I}\ ¡} \max_{1£\ i£\ n}\left\|F_i(t)F_i(t)Y_i^*(t)\right\|$$

or

$$\sup_{\substack{t\hat{I}\ ¡}} \max_{\substack{1£\ i£\ n \\ 1£\ j£\ n}}\left\|F_{ij}(t)T_{ij}(t)Y_{ij}^*(t)\right\| £\ K_0 \sup_{\substack{t\hat{I}\ ¡}} \max_{\substack{1£\ i£\ n \\ 1£\ j£\ n}}\left\|F_{ij}(t)F_{ij}(t)Y_{ij}^*(t)\right\| \tag{3.3}$$

For, define the mapping $R: D \circledR\ \mathscr{B}$ as $RT = T' - AT - TB$. Clearly, R is linear and bounded with $\|R\| £\ 1$. Let $RT = 0$, and the fact T satisfies the differential equation

$T' = AT + TB$ and hence $T \hat{I}\ \mathscr{B}$. This shows that T is $(F,\ Y)$-bounded on $¡$ of the system (1.1). Then

$$T(0)\hat{I}\ X_0\ Ç(X_-\ Å\ X_+) = \{0\}.$$

Thus, $T = 0$ so that R is one-to-one. To prove that R is "onto", for any $F \in \mathscr{B}$, let T be a $(F,\ Y)$-bounded on $¡$ of the system (1.1) and T be the solution of the Cauchy problem

$$T' = A(t)T + TB(t) + F(t)$$

satisfying $T(0) = (P_- + P_+)T(0)$. Then, $U = R - T$ is a solution of the system (3.2) with $U(0) = R(0) - (P_- + P_+)T(0)$. Thus, $U\hat{I}\ D$ and $RT = F$. Consequently, the mapping R is a bounded, one-to-one linear operator from one Banach space $\mathscr{B}$ to another Banach space B. Hence, R^{-1} exists and bounded, where

$$\left\|R^{-1}F\right\|_{\mathscr{B}} £\ \left\|R^{-1}\right\|\|F\|_{\mathscr{B}}\ \text{for all } F \in \mathscr{B}$$

It follows that

$$\left\|R^{-1}F\right\| = \left(\left\|R^{-1}\right\| - 1\right)\|F\| £\ K_0\|F\|_{\mathscr{B}}$$

where $K_0 = \left\| R^{-1} \right\| - 1$, which is equivalent to (3.3).

Let q_1 and q_2 be any fixed real numbers such that $q_1 < 0 < q_2$ and $F : \mathbb{R} \to \mathbb{R}^{n^2}$ be a function in $\mathscr{B}$ which vanishes on $(-\yen, q_1] \dot{\mathrm{E}} [q_2, \yen)$. Then it is easy to see that the function $T : \mathbb{R} \to \mathbb{R}^{n^2}$ defined as

$$
T(t) = \begin{cases}
-\displaystyle\int_{\theta_1}^{0} \phi(t)P_0\phi^{-1}(\sigma(s))f(s)ds - \int_{\theta_1}^{\theta_2} \phi(t)P_+\phi^{-1}(\sigma(s))f(s)\Delta s, & t < \theta_1 \\[3mm]
-\displaystyle\int_{\theta_1}^{t} \phi(t)P_-\phi^{-1}(\sigma(s))f(s)\Delta s + \int_{0}^{t} \phi(t)P_0\phi^{-1}(\sigma(s))f(s)\Delta s \\[3mm]
-\displaystyle\int_{\theta_1}^{\theta_2} \phi(t)P_-\phi^{-1}(\sigma(s))f(s)\Delta s, , & \theta_1 \le t \le \theta_2 \\[3mm]
\displaystyle\int_{0}^{\theta_2} \phi(t)P_+\phi^{-1}(\sigma(s))f(s)\Delta s, & t > \theta_2
\end{cases}
$$

is the solution in D of the system (1.1). Now if we put

$$
G(t,s) = \begin{cases}
\mathrm{F}(t)\mathrm{Y}^*(t)P_-\ \mathrm{Y}^{*^{-1}}(s)\mathrm{F}^{-1}(s), & s \pounds\ 0 < t \\[2mm]
\mathrm{F}(t)\mathrm{Y}^*(t)(P_0 + P_+)\mathrm{Y}^{*^{-1}}(s)\mathrm{F}^{-1}(s), & 0 < s < t \\[2mm]
-\ \mathrm{F}(t)\mathrm{Y}^*(t)P_+\mathrm{Y}^{*^{-1}}(s)\mathrm{F}^{-1}(s), & 0 < t \pounds\ s \\[2mm]
\mathrm{F}(t)\mathrm{Y}^*(t)P_-\ \mathrm{Y}^{*^{-1}}(s)\mathrm{F}^{-1}(s), & s < t \pounds\ 0 \\[2mm]
-\ \mathrm{F}(t)\mathrm{Y}^*(t)(P_0 + P_+)\mathrm{Y}^{*^{-1}}(s)\mathrm{F}^{-1}(s), & t \pounds\ s < 0 \\[2mm]
-\ \mathrm{F}(t)\mathrm{Y}^*(t)P_+\mathrm{Y}^{*^{-1}}(s)\mathrm{F}^{-1}(s), & t \pounds\ 0 \pounds\ s
\end{cases}.
$$

Then, G is continuous on $\mathbb{R}^{n^2}$ at all points except at $t = s$, and at $t = s$ G has a jump discontinuity of unit-magnitude (I_n). Then, we have

$$
T(t) = \grave{\mathrm{O}}_{q_1}^{q_2} G(t,s)F(s)ds \quad \text{for } t \hat{\mathrm{I}}\ \mathbb{R}.
$$

Indeed, for $q_1 > t$, we have

$$\int_{q_1}^{q_2} G(t,s)F(s)ds = -\int_{q_1}^{0} F(t)Y^*(t)(P_0 + P_+)Y^{*\,-1}(s)F^{-1}(s)F(s)ds -$$
$$\int_{0}^{q_2} F(t)Y^*(t)P_+ Y^{*\,-1}(s)F^{-1}(s)F(s)ds .$$

Rewrite the second integral as $\int_{0}^{q_1}...ds + \int_{q_1}^{q_2}...ds$, we get

$$\int_{q_1}^{q_2} G(t,s)F(s)ds = -\int_{q_1}^{0} F(t)Y^*(t)P_0 Y^{*\,-1}(s)F^{-1}(s)F(s)ds -$$
$$\int_{q_1}^{q_2} F(t)Y^*(t)P_+ Y^{*\,-1}(s)F^{-1}(s)F(s)ds = T(t) .$$

For $t \in [q_1,0)$, we have

$$\int_{q_1}^{q_2} G(t,s)F(s)ds = -\int_{q_1}^{t} F(t)Y^*(t)P_- Y^{*\,-1}(s)F^{-1}(s)F(s)ds -$$
$$\int_{t}^{0} F(t)Y^*(t)(P_0 + P_+)Y^{*\,-1}(s)F^{-1}(s)F(s)ds -$$
$$\int_{0}^{q_2} F(t)Y^*(t)P_+ Y^{*\,-1}(s)F^{-1}(s)F(s)ds =$$
$$\int_{q_1}^{t} F(t)Y^*(t)P_- Y^{*\,-1}(s)F^{-1}(s)F(s)ds +$$
$$\int_{0}^{t} F(t)Y^*(t)P_0 Y^{*\,-1}(s)F^{-1}(s)F(s)ds -$$
$$\int_{t}^{q_2} F(t)Y^*(t)P_1 Y^{*\,-1}(s)F^{-1}(s)F(s)ds = T(t) .$$

For $t \in (0,q_2)$, we have

$$\int_{q_1}^{q_2} G(t,s)F(s)ds = \int_{q_1}^{0} F(t)Y^*(t)P_- Y^{*\,-1}(s)F^{-1}(s)F(s)ds +$$
$$\int_{0}^{t} F(t)Y^*(t)(P_0 + P_-)Y^{*\,-1}(s)F^{-1}(s)F(s)ds -$$
$$\int_{t}^{q_2} F(t)Y^*(t)P_+ Y^{*\,-1}(s)F^{-1}(s)F(s)ds =$$
$$\int_{q_1}^{t} F(t)Y^*(t)P_- Y^{*\,-1}(s)F^{-1}(s)F(s)ds +$$
$$\int_{0}^{t} F(t)Y^*(t)P_0 Y^{*\,-1}(s)F^{-1}(s)F(s)ds -$$
$$\int_{t}^{q_2} F(t)Y^*(t)P_+ Y^{*\,-1}(s)F^{-1}(s)F(s)ds = T(t) .$$

For $t > q_2$, we can easily show that $\int_{q_1}^{q_2} G(t,s)F(s)ds = T(t)$. Therefore,

$$\sup_{t \in i}\left\| F(t)Y^*(t)\int_{q_1}^{q_2} G(t,s)F(s)ds \right\| \le K\int_{q_1}^{q_2} \left\| F(t)Y^*(t)F(t) \right\| dt$$

for all $t \in \mathbb{R}$. Hence,

$$\left\| F(t)Y^*(t)G(t,s)Y^{*-1}(s)F^{-1}(s) \right\| \leq K \quad \text{for all } t \in \mathbb{R} .$$

Now, to prove the converse statement, suppose the fundamental matrices of Y and Z of (1.2) and (1.3) satisfy the condition (3.1) for some $K > 0$. Let $F : \mathbb{R} \to \mathbb{R}^{n^2}$ be a Lebesgue (F, Y)-delta integrable function on $\mathbb{R}$. We consider the function $U : \mathbb{R} \to \mathbb{R}^{n^2}$ defined by

$$
\begin{aligned}
U(t) = & \int_{-\infty}^{\infty} F(t)Y^*(t)Y^{*-1}(s)F^{-1}(s)T(s)ds + \\
& \int_0^t F(t)Y^*(t)Y^{*-1}(s)F^{-1}(s)T(s)ds - \\
& \int_t^{\infty} F(t)Y^*(t)Y^{*-1}(s)F^{-1}(s)T(s)ds .
\end{aligned}
\tag{3.3}
$$

Then, the function is well defined on $\mathbb{R}$, and

$$\left\| F(t)Y^*(t)U(t) \right\| \leq K \int_{-\infty}^{\infty} \left\| F(s)Y^*(s)T(s) \right\| ds ,$$

which shows that U is (F, Y)-bounded on $\mathbb{R}$. Hence the proof is complete.   □

Theorem 3.2: If the homogeneous Sylvester system (1.1) has no non-trivial (F, Y)-bounded solution on $\mathbb{R}$, then (1.1) has a unique (F, Y)-bounded solution on $\mathbb{R}$ for every Lebesgue (F, Y)-integrable function $F : \mathbb{R} \to \mathbb{R}^{n^2}$ if and only there exists a $K > 0$ such that

$$\left\| F(t)Y(t)Z^*(t)Y^*(t)P_- Y^{*-1}(s)Z^{*-1}(s)Y^{-1}(s)F^{-1}(s) \right\| \leq K_s \quad \text{for } -\infty < s < t < \infty$$

and

$$\left\| F(t)Y(t)Z^*(t)Y^*(t)P_+ Y^{*-1}(s)Z^{*-1}(s)Y^{-1}(s)F^{-1}(s) \right\| \leq K_t \quad \text{for } -\infty < t < s < \infty$$

The proof follows by taking $P_0 = 0$ in Theorem 3.1.

Theorem 3.3: Suppose that a fundamental matrix $Y(t)$ of $T' = AT$ and a fundamental matrix $Z(t)$ of $T' = B^*T$ satisfy the conditions:

(i) $\left\| F(t)Y(t)Z^*(t)Y^*(t)P_- Y^{*-1}(s)Z^{*-1}(s)Y^{-1}(s)F^{-1}(s) \right\| \leq K$ for $t > 0, s \leq 0$

$\left\| F(t)Y(t)Z^*(t)Y^*(t)(P_0 + P_-)Y^{*-1}(s)Z^{*-1}(s)Y^{-1}(s)F^{-1}(s) \right\| \leq K$ for $t > 0, s > 0, s < t$

$$\left\| F(t)Y(t)Z^*(t)Y^*(t)P_+ Y^{*-1}(s)Z^{*-1}(s)Y^{-1}(s)F^{-1}(s)\right\| \pounds\ K \ \text{ for } t>0, s>0, s>t$$

$$\left\| F(t)Y(t)Z^*(t)Y^*(t)P_- Y^{*-1}(s)Z^{*-1}(s)Y^{-1}(s)F^{-1}(s)\right\| \pounds\ K \ \text{ for } t \pounds\ 0, s<t$$

$$\left\| F(t)Y(t)Z^*(t)Y^*(t)(P_0 + P_-) Y^{*-1}(s)Z^{*-1}(s)Y^{-1}(s)F^{-1}(s)\right\| \pounds\ K \ \text{ for } t \pounds\ 0, s>t, s<0$$

$$\left\| F(t)Y(t)Z^*(t)Y^*(t)P_+ Y^{*-1}(s)Z^{*-1}(s)Y^{-1}(s)F^{-1}(s)\right\| \pounds\ K \ \text{ for } t \pounds\ 0, s>0, s³\ t$$

(ii) $\displaystyle\lim_{t® ¥}\left\| F(t)Y(t)Z^*(t)Y^*(t)P_0\right\| = 0$

$\displaystyle\lim_{t® ¥}\left\| F(t)Y(t)Z^*(t)Y^*(t)P_1\right\| = 0$

$\displaystyle\lim_{t® ¥}\left\| F(t)Y(t)Z^*(t)Y^*(t)P_-\right\| = 0$

and

(iii) the function $F : ¡\ ®\ ¡^{n^2}$ is Lebesgue-delta integrable on $¡$.

Then, every (F, Y)-bounded solution T of (1.1) is such that

$$\lim_{t® ± ¥}\left\| F(t)Y(t)Z^*(t)Y^*(t)T(t)\right\| = 0 .$$

The proof is similar to that of the Theorem 3.3 in [4].

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